Quantum Field Theory

Set 8

Exercise 1: Application of Lippman-Schwinger equation

Starting from the usual Lippman-Schwinger equation,

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H_0 + i\epsilon} H_I |\psi_\alpha^+\rangle,$$

show that it can be written in the equivalent form

$$|\psi_\alpha^+\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - H + i\epsilon} H_I |\phi_\alpha\rangle.$$

Use this formula to deduce the T-matrix element $T_{\beta\alpha} \equiv \langle\phi_\beta|H_I|\psi_\alpha^+\rangle \equiv \langle\phi_\beta|T|\phi_\alpha\rangle$ in the case in which $H_I = V_1(\vec{x}) + V_2(\vec{x})$, with $V_2(\vec{x}) = V_1(\vec{x} + \vec{A}) \equiv e^{i\vec{P}\cdot\vec{A}} V_1(\vec{x}) e^{-i\vec{P}\cdot\vec{A}}$, where $V_1(\vec{x})$ is significantly different from 0 only in a small region around a given point $\vec{x}_0$, and the distance $|\vec{A}|$ between the two potentials is much larger than the linear size of that region. Show that in this limit the T-matrix splits up in two independent pieces, $T = T_1(\vec{x}) + e^{i\vec{P}\cdot\vec{A}} T_1(\vec{x}) e^{-i\vec{P}\cdot\vec{A}}$.
Generalize this formula to an interaction Hamiltonian made of a set of N potentials $V_j(\vec{x})$ (all mutually far apart), and to the continuum case.

Exercise 2: Scattering from a general potential

Consider a state $|\phi_k\rangle$ which is an eigenstate of a free Hamiltonian H_0. For simplicity let us consider $H_0 = \frac{p^2}{2m}$. Let us assume that at a certain finite time t and a finite distance L the states start interacting with a potential V. The system is now described by the full Hamiltonian $H = H_0 + V$. We also assume that the interaction with the potential is localized in time and space, so that the system, far away from the interaction point and after enough time, can still be described in terms of eigenstates of H_0.
Recalling the Lippmann-Schwinger equation,

$$|\Psi_k^\pm\rangle = |\phi_k\rangle + \frac{1}{E_k - H_0 \pm i\epsilon} H_I |\Psi_k^\pm\rangle, \tag{1}$$

extract an expression for the asymptotic states for this framework and show that at distances much larger than the typical size of the interaction it holds:

$$\Psi_k^\pm(x) = e^{i\vec{k}\cdot\vec{x}} + \frac{e^{\pm ikr}}{r} f(\hat{x}, k). \tag{2}$$

Exercise 3: Differential cross section $2 \to 2$ in the center of mass

Consider a scattering process $AB \to CD$ where the particles have four-momenta satisfying $P_A + P_B = P_C + P_D$. The differential cross section reads

$$d\sigma_{AB\to CD} = \frac{1}{4E_A E_B |\vec{v}_A - \vec{v}_B|} |\mathcal{M}_{AB\to CD}|^2 \, d\Phi_2\,,$$

where the 2-bodies phase space is

$$d\Phi_2 = \frac{d^3p_C}{(2\pi)^3 2E_C} \frac{d^3p_D}{(2\pi)^3 2E_D} (2\pi)^4 \delta^4 \left(P_A + P_B - P_C - P_D\right).$$

Show that for particles colliding along the $\hat{z}$ axis the flux factor can be written as

$$E_A E_B |v_A^z - v_B^z| = \sqrt{(P_A^\mu P_{B\mu})^2 - m_A^2 m_B^2}.$$

If we are dealing with scalar particles in the initial and final state or we sum and average over all possible polarizations of non-scalar particles the Lorentz invariant matrix element $|\mathcal{M}_{AB\to CD}|^2$ can only depend on scalar combinations of the four momenta P_i. In general it could depend also on the polarizations of initial or final states. Introduce thus the *Mandelstam variables*

$$s = (P_A + P_B)^2 = (P_C + P_D)^2, \qquad t = (P_A - P_C)^2 = (P_B - P_D)^2, \qquad u = (P_A - P_D)^2 = (P_B - P_C)^2,$$

and compute the two bodies phase space and the differential cross section in the center of mass frame. In doing that, show that the scattering process has two degrees of freedom, which can be chosen as the polar and azimuthal angles of the scattering products w.r.t. the incoming particles. Express the phase space, and consequently the differential cross section, in terms of these variables, the masses, and the Mandelstam invariant s.
Write down the expression for the cross section in the particular cases $m_C = m_D = m$ or $m_C = 0$, $m_D = m$.

Exercise 4 (optional): Asymptotic states in Quantum Mechanics

Consider a one dimensional quantum system with a potential which is significantly different from zero only in a region $x \in [-L, L]$ and is rapidly decreasing outside. Call $\psi_k(x)$ the solution (of the Schroedinger equation for the interacting theory) corresponding to an energy $E_k = \frac{k^2}{2m}$. In the region where the potential is approximately zero we can write a particular solution with energy E_k as

$$\psi_k^+(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x \ll -L, \\ Te^{ikx}, & x \gg L, \end{cases}$$

where R and T are coefficients. Consider now a wave packet $\psi^+(x)$ which is a (narrow) gaussian superposition of particular solutions $\psi_k^+(x)$ around a momentum p. Evolve $\psi^+(x)$ in the past ($t \to -\infty$) and show that the result describes a free wave-packet incoming from the left. In your calculations neglect the spread of the wavepacket, that would otherwise be completely spread out in the $t \to \pm\infty$ limits. Conclude that $\psi^+(x)$ is the in- asymptotic state.
Consider now another particular solution

$$\psi_k^-(x) = \begin{cases} e^{ikx} + R'e^{-ikx}, & x \gg L, \\ T'e^{ikx}, & x \ll -L, \end{cases}$$

and show analogously that the gaussian packet $\psi^-(x)$ is the out- asymptotic state, namely at $t \to \infty$ it describes a free packet escaping towards the right.